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"AFFINE" HECKE ALGEBRAS ASSOCIATED TO KAC-MOODY GROUPS.

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In this paper we give a geometric construction of Cherednik's double affine Hecke algebra [C]. This construction is suitable for producing a certain class of modules, and in a paper in preparation we will use the results of this paper to describe these modules.

The construction is modeled on the fantastic construction of the usual affine Hecke algebra as the equivariant K-theory of the Steinberg variety [KL1], i.e. the Lagrangian subvariety Λ of $T^*(\mathcal{B} \times \mathcal{B})$ given by the union of the conormal bundles to the G-orbits on $\mathcal{B} \times \mathcal{B}$. As such, it is "morally obvious" what to do here—one just replaces G and $\mathcal{B}$ by the corresponding affine group and its flag variety.

At this level, the idea that a result similar to our theorem 4.5 must be true was obvious to a lot of people. In particular, Cherednik, Ginzburg, Kazhdan and Lusztig have told us they knew this, and we believe that Kashiwara and Tanisaki must also have known.

Thus the main content is to actually *do* this for the infinite dimensional G and $\mathcal{B}$. Here again, we can claim no great originality. The beautiful paper [KT] gives a lovely meaning to $\mathcal{D}$-modules on $\mathcal{B}$. Using this, our main contribution is to define what coherent sheaves on $T^*\mathcal{B}$ are. We do this using the technique of [Gr].

The fundamental importance of Cherednik's algebra suggests that it is worth publishing this note. In particular, the lovely structure of the modules we have found, as well as the conjectures (work in progress) based on this perhaps justify it. (Not to mention the remarkable work of Cherednik!)

We would like to mention just one observation, connected with several projects of ours. That is, that from the point of view of this paper, Cherednik's 'Fourier transform' is highly non-trivial.

1. Notation

We will keep notation as in [KT], and suggest the reader have a copy at hand.

Throughout this paper, $\mathfrak{g}$ will be a Kac-Moody algebra with symmetrisable generalised Cartan matrix $(a_{ij})_{i,j\in I}$, and G will be an associated group as in [K]. We will write T for the maximal torus of G, X for its lattice of characters, and $\mathcal{B} = G/B$ for the flag variety. All other notation will be as in [KT], and the definition of these objects are those of [K,KT].

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

To this data of $(T, \mathfrak{g})$ we can associate two algebras: the Hecke algebra H and the "affine" Hecke algebra $\widehat{H}$. Recall that H is the free $\mathbf{Z}[q, q^{-1}]$-module with basis T_w, $w \in W$, and multiplication defined by

$$(T_s + 1)(T_s - q) = 0, \qquad l(s) = 1$$

$$T_s T_w = T_{sw}, \qquad l(sw) = l(s) + l(w).$$

Also recall that $\widehat{H} = H \otimes_{\mathbf{Z}} \mathbf{C}[X]$, and that this is given the unique structure of an algebra such that the Hecke algebra H and the group algebra $\mathbf{C}[X]$ are subalgebras, and

$$f T_s - T_s {}^s f = (q-1) \frac{f - {}^s f}{1 - e^{-\alpha}}$$

where $s = s_\alpha$ is a simple reflection through α and e^α is the corresponding element of $\mathbf{C}[X]$.

In the case G is of finite type, this is the definition of $\widehat{H}$ due to Bernstein and Zelevinski. Cherednik's double affine Hecke algebra [C] is also of this type. It occurs when G is an affine Kac-Moody group, with a torus of dimension $l+1$ which contains the "degree operator" but not the center. Here l is the rank of the finite reductive group $\bar{G}$, and G is the loop group of $\bar{G}$.

In particular, for G of affine type with torus of dimension $l+2$, the algebra we have just defined is a generalisation of Cherednik's algebra to arbitrary central charge. This seems to be new.

2. The Hecke algebra H

This section is a minor addition to [KT].

Let P_i, $i \in I$, be the parabolic subgroup of G, defined as in [KT], and let ${}_i\mathcal{P} = G/P_i$ be the associated generalised flag variety, defined as in [K].

The we have a map of schemes, $r_i : \mathcal{B} \to {}_i\mathcal{P}$, which is a $\mathbf{P}^1$ fibration. The precise statement we need is

Lemma 2.1. *Let F be a finite subset of W such that $s_i F \subseteq F$, $Y = \mathcal{B}^F$ is open, $Z = \mathcal{B}_F$ is closed and $Z \subseteq Y$. Define ${}_iY = r_i(Y)$. Then for sufficently large k*

i) The group U_k^- acts on ${}_iY$ locally freely, and the quotient is a finite dimensional smooth variety.

ii) The induced morphism $U_k^- \backslash Y \to U_k^- \backslash {}_iY$ is a $\mathbf{P}^1$-bundle fibration.

iii) The natural map $r_i(Z) \to U_k^- \backslash {}_iY$ is a closed immersion.

iv) The diagram

$$\begin{array}{ccc} Y_a & \longrightarrow & Y_b \\ \big\downarrow & & \big\downarrow \\ {}_iY_a & \longrightarrow & {}_iY_b \end{array}$$

is cartesian.

We write ${}_iY_a = U_a^- \backslash {}_iY$, $Y_a = U_a^- \backslash Y$. Also denote the natural map of (ii) as ${}_ir_l : Y_l \to {}_iY_l$, and write (as in [KT]) $p_a : {}_iY \to {}_iY_a$, $p_b^a : {}_iY_a \to {}_iY_b$, $a \geq b \geq k$ for the natural morphisms. These are affine. We call such data (Z, Y, k) a P_i-stable admissible triple.

Now, let (Z, Y, k) be a P_i-stable admissible triple, and $\mathbf{H}(Z, Y)$ be the semisimple category of pure mixed Hodge (right) modules, defined as in [KT,2.2.11] for $\lambda = 0$. (In other words, use pure mixed Hodge modules [S] rather than $\mathcal{D}$-modules, and write $\mathbf{H}(Z, Y)$ for what is denoted $\mathbf{H}(0, Z, Y)$ in [KT,2.2.11]).

Let $\mathcal{A}_l$ be a pure mixed Hodge module on Y_l with support contained in Z. Then define $C_i'\mathcal{A}_l$ as the sum

$$\oplus_j {}^p H^j(({}_i r_l)^*({}_i r_l)_*\mathcal{A}_l)$$

where $({}_i r_l)^*$, $({}_i r_l)_*$ denote the usual derived functors of mixed Hodge modules as in [S]. Note that by the decomposition theorem, $({}_i r_l)^*({}_i r_l)_*\mathcal{A}_l$ is semisimple and pure of weight α if $\mathcal{A}_l$ is pure of weight α, and so $C_i'\mathcal{A}_l$ is a split semisimple pure MHM. (Be warned that ${}^pH^0 f_*$ is what is denoted $\int_f$ in [KT]).

Lemma 2.2. *If (Z, Y) is P_i-stable admissible, then if $\mathcal{A} \in \mathbf{H}(Z, Y)$ we have $C_i'\mathcal{A} \in \mathbf{H}(Z, Y)$ also.*

Here, if $\mathcal{A} = (\mathcal{A}_a, \gamma_b^a)$, then $C_i'\mathcal{A} = (C_i'\mathcal{A}_a, \tilde{\gamma}_b^a)$, and $\tilde{\gamma}_b^a$ are induced from γ_b^a and the base change isomorphism applied to the Cartesian diagram of lemma 2.1. The proof is immediate (using the fact that the horizontal maps are affine space fibrations, and so $\int_{p_b^a} = (p_b^a)_*$).

Write $\mathbf{H} = \lim_{\xrightarrow{Z}} \lim_{\xleftarrow{Y}} \mathbf{H}(Z, Y)$ as in [KT,2.2]. Then $\mathbf{H}$ is a semisimple abelian category, where every object has finite length, and $\mathbf{H}$ is a free $\mathbf{Z}[q, q^{-1}]$ module with basis C_w', $w \in W$. Here, q acts by Tate twist, and C_w' represents the pure simple Hodge module which is the perverse extension of the constant local system on $\mathcal{B}_w$ with weight $l(w)$.

Proposition 2.3. *The operators $C_i' : \mathbf{H} \to \mathbf{H}$ give $\mathbf{H}$ the structure of the regular H-module, where C_i' acts as multiplication by $T_{s_i} + 1$ on $\mathbf{H}$.*

The proof is standard and omitted. (It consists of enlarging $\mathbf{H}$ to allow non-semisimple perverse sheaves, so that the constant local systems on $\mathcal{B}_w$ extended by 0, call them T_w, are in this new category $\tilde{\mathbf{H}}$. Then the Grothendieck group of $\tilde{\mathbf{H}}$ is the same as that of $\mathbf{H}$, and the elements T_w clearly satisfy $T_s T_w = T_{sw}$ if lengths add, and $(T_s + 1)^2$ is as claimed by standard properties of $\mathbf{P}^1$ fibrations.)

3. Definition of $\mathrm{Coh}^{T \times \mathbf{C}^*}(\Lambda)$

3.1

Let X be a variety on which our torus T acts. A T-equivariant coherent sheaf on X is a coherent sheaf on the quotient stack X/T; explicitly it is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a coherent sheaf on X and $\phi : p^*\mathcal{A} \to a^*\mathcal{A}$ is an isomorphism satisfying the usual compatibilities. Here $p : T \times X \to X$, $(t, x) \mapsto x$ and $a : T \times X \to X$, $(t, x) \mapsto tx$ are the projection and action maps.

Denote the category of T-equivariant coherent sheaves $\mathrm{Coh}^T(X)$. This is an exact category, write $D^T(\mathcal{O}_X)$ for its derived category and $K^T(X)$ for the Grothendieck group of $\mathrm{Coh}^T(X)$ which is canonically isomorphic to the Grothendieck group of $D^T(\mathcal{O}_X)$.

If $f : X \to Z$ is a morphism of T-spaces, we denote by f^* the left derived functor of the usual pullback of $\mathcal{O}$-modules, and by f_* the right derived functor of the usual pushforward of $\mathcal{O}$-modules. Recall that if f is a regular morphism, $f^*\mathcal{A}$ is a finite complex of coherent $\mathcal{O}$-modules if $\mathcal{A}$ is, and so $f^* : D^T(\mathcal{O}_Z) \to D^T(\mathcal{O}_X)$ in this

case, and if the restriction of f to the support of $\mathcal{A}$ is proper, then $f_*\mathcal{A}$ is a finite complex of coherent $\mathcal{O}$-modules if $\mathcal{A}$ is, so $f_* : D^T(\mathcal{O}_X) \to D^T(\mathcal{O}_Z)$ in this case. We refer to [T] for details.

Now let $Y \subseteq X$ be a closed T-stable subvariety. Write $D^T(\mathcal{O}_X, Y)$ for the triangulated subcategory of $D^T(\mathcal{O}_X)$ consisting of sheaves whose cohomology has support in Y, and write $K^T(X,Y)$ for its Grothendieck group.

If $i : Y \hookrightarrow X$ denotes the inclusion, then $i_* : D^T(\mathcal{O}_Y) \to D^T(\mathcal{O}_X, Y)$ induces an isomorphism $i_* : K^T(Y) \to K^T(X,Y)$. In particular, $K^T(X,Y)$ does not depend on the ambient variety X. (We recall that a sheaf on X with support on Y need not be of the form $i_*\mathcal{A}$, and so there is a small subtlety in these statements).

3.2

Let (Z,Y,k) be an admissible triple. Then $Z = \mathcal{B}_F$, for F a finite subset of W. Define, for $l \geq k$ and $F' \subseteq F$

$$\Lambda^l_{F'} = \bigcup_{w \in F'} T^*_{\mathcal{B}_w} Y_l \subseteq T^* Y_l.$$

Write $\Lambda^l_Z = \Lambda^l_F$, and note that the group $T \times \mathbf{C}^*$ acts on $\Lambda^l_{F'}$.

Now, the affine fibration $p^a_b : Y_a \to Y_b$ gives rise to a correspondence

$$T^*Y_a \xleftarrow{\alpha^a_b} Y_a \times_{Y_b} T^*Y_b \xrightarrow{\beta^a_b} T^*Y_b$$

where we suppose $a \geq b \geq k$.

Ignore the T-action for a moment. Then, choosing a small open set $U_b \subseteq Y_b$, we get an open set $U_a = (p^a_b)^{-1}(U_b)$ of Y_a, and we can identify

$$U_a \simeq U_b \times \mathbf{C}^n$$

where $p^a_b : U_a \to U_b$ is projection onto the second factor, and $Z \cap U_a \hookrightarrow U_a$ is the map $z \mapsto (z, 0)$.

It follows that $U_a \times_{U_b} T^*U_b \simeq T^*U_b \times \mathbf{C}^n \times 0$, $T^*U_a \simeq T^*U_b \times \mathbf{C}^n \times \mathbf{C}^n$, and that α^a_b is the obvious embedding, β^a_b the evident projection, and that $\Lambda^a_Z \cap T^*U_a \simeq (\Lambda^b_Z \cap T^*U_b) \times 0 \times \mathbf{C}^n$.

In particular, we get $(\alpha^a_b)^{-1}(\Lambda^a_Z) \simeq \Lambda^b_Z \times 0 \times 0$, and β^a_b induces an isomorphism of this onto Λ^b_Z.

Thus, if we denote by $I^a_b : D(\mathcal{O}_{T^*Y_a}, \Lambda^a_Z) \to D(\mathcal{O}_{T^*Y_b}, \Lambda^b_Z)$ the map $(\beta^a_b)_*(\alpha^a_b)^*$, then this does take a coherent sheaf on T^*Y_a supported on Λ^a_Z to a complex of sheaves on T^*Y_b, with cohomology supported on Λ^b_Z.

Further, the induced map $I^a_b : K(T^*Y_a, \Lambda^a_Z) \to K(T^*Y_b, \Lambda^b_Z)$ is an isomorphism (as restricting to the zero section of a vector bundle induces an isomorphism in K-theory).

Now, recall that we have a T-action. We still get maps

$$K^{T\times\mathbf{C}^*}(\Lambda^a_Z) \xrightarrow{\sim} K^{T\times\mathbf{C}^*}(T^*Y_a, \Lambda^a_Z) \xrightarrow{I^a_b} K^{T\times\mathbf{C}^*}(T^*Y_b, \Lambda^b_Z) \xleftarrow{\sim} K^{T\times\mathbf{C}^*}(\Lambda^b_Z)$$

and in fact

Lemma 3.3. *$I^a_b : K^{T\times\mathbf{C}^*}(T^*Y_a, \Lambda^a_Z) \to K^{T\times\mathbf{C}^*}(T^*Y_b, \Lambda^b_Z)$ is an isomorphism.*

But this follows from the previous discussion, either by restricting to T-stable neighbourhoods U_b of Y_b and observing we still have a splitting (Sumihoro's theorem), or by restricting to T-fixpoints and using the embedding $K^T(X) \hookrightarrow K^T(X^T)$ and the preceding arguments.

Now we can define a group depending on (Y, Z, k) as the limit of these isomorphisms; i.e. an element here is a sequence $x_a \in K^{T\times\mathbf{C}^*}(T^*Y_a, \Lambda^a_Z)$, $a \geq k$ such that $I^a_b x_a = x_b$, $a \geq b$. We can take the limit over k. Further, if Y' is an admissible open set containing Y, then the group defined for (Y, Z) is canonically isomorphic to that for (Y', Z). So we can take the limit over admissible open sets Y containing Z. Call this resulting group

$$K^{T\times\mathbf{C}^*}(\Lambda_Z).$$

Finally, if Z' is an admissible closed set containing Z, the the closed embeddings $i_a : \Lambda^a_Z \hookrightarrow \Lambda^a_{Z'}$ give rise to an embedding $i_* : K^{T\times\mathbf{C}^*}(\Lambda_Z) \hookrightarrow K^{T\times\mathbf{C}^*}(\Lambda_{Z'})$. Take the limit over admissible closed sets Z to get (compare [KT,2.2])

$$K^{T\times\mathbf{C}^*}(\Lambda) = \varinjlim_Z K^{T\times\mathbf{C}^*}(\Lambda_Z).$$

3.4 This Grothendieck group is somewhat crude. However, the same procedure works nicely to define $\mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda)$, "actual" coherent sheaves with finite support on $\mathcal{B}$ and finite cosupport along the cotangent directions in $T^*\mathcal{B}$ (and supported in Λ).

The point is to define for each admissible triple (Y, Z, k) a "coherent sheaf" $\mathcal{A}$ as a family $\mathcal{A} = ((\mathcal{A}_l)_{l\geq k}, (\gamma^a_b)_{a\geq b\geq k})$, where $\mathcal{A}_l$ is a coherent sheaf on T^*Y_l supported on Λ^l_Z and

$$\gamma^a_b : I^a_b\mathcal{A}_a \xrightarrow{\sim} \mathcal{A}_b$$

is an isomorphism satisfying the chain condition $\gamma^a_c I^a_b = \gamma^b_c I^b_c \gamma^a_b$. Note that in order for γ^a_b to exist, $\mathcal{A}_a$ must be smooth (locally constant) in the directions transverse to the bundle embedding $Y_a \times_{Y_b} T^*Y_b \hookrightarrow T^*Y_a$, otherwise $I^a_b\mathcal{A}_a$ will be a complex of sheaves.

Nonetheless, enough such families exist; one can define morphisms in the obvious way, and take limits as above, getting the abelian category $\mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda)$. Then the Grothendieck group of $\mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda)$ is $K^{T\times\mathbf{C}^*}(\Lambda)$, as defined above.

We will not need this precision, but it's nice to know its there.

4. Action of $\widehat{H}$ on $K^{T\times\mathbf{C}^*}(\Lambda)$.

4.1

We now define an action of the Hecke algebra H on $K^{T\times\mathbf{C}^*}(\Lambda)$.

Let (Z, Y, k) be a P_i-stable admissible triple, $i \in I$. Let $a \geq b \geq k$. Define

$${}_i\mathcal{R}_a = \{(x, y) \in Y_a \times Y_a \mid {}_ir_a(x) = {}_ir_a(y)\},$$

and let π_a, π'_a denote the first and second projections. Then by lemma 2.1, $\pi_a : {}_i\mathcal{R}_a \to Y_a$ is a $\mathbf{P}^1$-bundle. We obtain an induced correspondence on cotangent bundles

$$T^*Y_a \xleftarrow{\pi_a} T^*_{{}_i\mathcal{R}_a}(Y_a \times Y_a) \xrightarrow{\pi'_a} T^*Y_a.$$

One can easily check that $\pi'_a(\pi_a^{-1}(\Lambda_Z^a)) \subseteq \Lambda_Z^a$, and so we get a map $\bar{C}'_i : D^{T\times\mathbf{C}^*}(\mathcal{O}_{T^*Y_a}, \Lambda_Z^a) \to D^{T\times\mathbf{C}^*}(\mathcal{O}_{T^*Y_a}, \Lambda_Z^a)$ defined by

$$\bar{C}'_i\mathcal{A} = (\pi'_a)_*(\pi_a^*\mathcal{A}\otimes^L\rho^{-1}\omega_{{}_i\mathcal{R}_a/Y_a})[1]$$

where $\omega_{{}_i\mathcal{R}_a/Y_a}$ is the bundle of relative top forms $\omega_{{}_i\mathcal{R}_a/Y_a} = \pi_a^*(\omega_{Y_a}^{\otimes^{-1}}) \otimes_{\pi_a^{-1}\mathcal{O}_{Y_a}} \omega_{{}_r\mathcal{R}_a}$, pulled back via $\rho : T^*_{{}_i\mathcal{R}_a}(Y_a \times Y_a) \to {}_i\mathcal{R}_a$. (Note that here we use the first projection, unlike in [Gr,2], as we are using right $\mathcal{D}$-modules).

Furthermore, one can easily check that if $\mathcal{A} \in D^{T\times\mathbf{C}^*}(\mathcal{O}_{T^*Y_a}, \Lambda_Z^a)$, $a \geq b \geq k$, the $\bar{C}'_i I_b^a\mathcal{A} = I_b^a\bar{C}'_i\mathcal{A}$; the diagram chase is omitted.

It follows we have well defined maps $\bar{C}'_i : K^{T\times\mathbf{C}^*}(\Lambda) \to K^{T\times\mathbf{C}^*}(\Lambda)$

4.2

Now let $w \in W$ be such that $\mathcal{B}_w \subseteq Z$. The pure mixed Hodge module which is the perverse extension of the constant local system on $\mathcal{B}_w$, with weight $l(w)$, is part of a compatible family of such, which we denoted C'_w in §2.

If $(M, F, \dots)$ represents this in Y_a, then $\mathrm{gr}_F M \in \mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda_Z^a)$, and by definition of $\mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda)$ and [Gr,2.1.2] these individual coherent sheaves patch togethor to to give an element of $\mathrm{Coh}^{T\times\mathbf{C}^*}(\Lambda)$. Denote this $\mathrm{gr}\, C'_w = \bar{C}'_w$.

It is clear that $\{\bar{C}'_w \mid w \in W\}$ are $\mathbf{Z}[q, q^{-1}]$-linearly independant in $K^{T\times\mathbf{C}^*}(\Lambda)$, and so we have defined an embedding $\mathrm{gr} : \mathbf{H} \to K^{T\times\mathbf{C}^*}(\Lambda)$.

Moreover, by the compatibility of gr and correspondences (see [Gr,2]), the action of the Hecke algebra H by right multiplication (by C'_i) induces an action on $\mathrm{gr}\mathbf{H}$; this is precisely the action by the operators $\bar{C}'_i$ defined above.

4.3

Let $\lambda : T \to \mathbf{C}^*$ be a character of T, $\mathcal{O}_{Y_l}(\lambda)$ the invertible line bundle on Y_l defined in [KT,2.2.5]. Then if $p : T^*Y_l \to Y_l$ denotes the canonical projection, we get an operator $\Theta_\lambda : D^{T\times\mathbf{C}^*}(\mathcal{O}_{T^*Y_a}, \Lambda_Z^a) \to D^{T\times\mathbf{C}^*}(\mathcal{O}_{T^*Y_a}, \Lambda_Z^a)$, $\mathcal{A} \mapsto \mathcal{A} \otimes p^*\mathcal{O}_{Y_l}(\lambda)$.

Clearly $\Theta_\lambda I_b^a = I_b^a\Theta_\lambda$, and $\Theta_\lambda\Theta_\mu = \Theta_{\lambda+\mu}$. This gives an action of $\mathbf{C}[X]$, the group algebra of X, on $K^{T\times\mathbf{C}^*}(\Lambda)$. Recall that W also acts on $\mathbf{C}[X]$.

Proposition 4.4. *If $f \in \mathbf{C}[X]$, and $s = s_\alpha$ is a simple reflection in W, then as operators on $K^{T\times\mathbf{C}^*}(\Lambda)$*

$$fT_s - T_s{}^sf = (q-1)\frac{f - {}^sf}{1 - e^{-\alpha}},$$

where $T_s = \bar{C}_s - 1$.

Let $Y \xrightarrow{\pi} \bar{Y}$ be a $\mathbf{P}^1$-fibration, $\mathcal{R} = \{(x, y) \in Y \times Y \mid \pi(x) = \pi(y)\}$, $T^*Y \leftarrow T^*_{\mathcal{R}}(Y \times Y) \to T^*Y$ the associated correspondence. Then the proposition is true in the generality of such correspondences. It is the semisimple rank 1 “affine” version of [KL1,1.3o2] and easily follows from standard facts about the cohomology of $\mathbf{P}^1$-bundles and the Koszul complex. We omit the short proof.

It follows that we have an action of $\widehat{H}$ on $\widehat{H}.\bar{C}'_1 \subseteq K^{T\times\mathbf{C}^*}(\Lambda)$.

Theorem 4.5. *The map $\widehat{H} \to K^{T\times\mathbf{C}^*}(\Lambda)$, $h \mapsto h.\bar{C}'_1$ is an isomorphism.*

Proof. Refine the Bruhat order $\leq$ on W to a total order on W, still denoted $\leq$. Write $x < y$ if $x \leq y$ and $x \neq y$. Now, if (Z, Y, k) is an admissible triple, $a \geq k$, $\mathcal{B}_w \subseteq Z$ then

$$K^{T\times\mathbf{C}^*}(\Lambda_w^a) = K^{T\times\mathbf{C}^*}(T^*_{\mathcal{B}_w}Y_a) = K^{T\times\mathbf{C}^*}(\mathcal{B}_w) = K^{T\times\mathbf{C}^*},$$

the last two equalities as $T^*_{\mathcal{B}_w} Y_a$ is an affine space bundle over $\mathcal{B}_w$, and $\mathcal{B}_w$ is itself affine. It follows [T] that we have a short exact sequence in K-homology

$$0 \to K^{T\times \mathbf{C}^*}(\Lambda^a_{<w}) \to K^{T\times \mathbf{C}^*}(\Lambda^a_{\leq w}) \xrightarrow{j^*} K^{T\times \mathbf{C}^*}(\Lambda^a_w) \to 0.$$

Now, if $(M, F, \dots)$ is the mixed Hodge module corresponding to the basis element $C'_z \in \mathbf{H}$, then it is clear that $\operatorname{gr} C'_z \in K^{T\times \mathbf{C}^*}(\Lambda^a_{<w})$ if $z < w$, and $j^*(\Theta_\lambda \operatorname{gr} C'_w), \lambda \in X$ form a basis of $K^{T\times \mathbf{C}^*}(\Lambda^a_w)$ over $\mathbf{Z}[q, q^{-1}]$.

As all these short exact sequences are compatible with the maps I^a_b, we see that we have produced a filtration of $K^{T\times \mathbf{C}^*}(\Lambda)$, and an isomorphism from $\widehat{H} \to \operatorname{gr} K^{T\times \mathbf{C}^*}(\Lambda)$ (as $\Theta_\lambda C'_w$, $\lambda \in X, w \in W$ form a basis of $\widehat{H}$). The theorem follows.

5. Standard modules.

Define $T^*\mathcal{B} = G \times^B \mathfrak{n} = \{(x, gB) \in \mathfrak{g} \times \mathcal{B} \mid g^{-1}xg \in \mathfrak{n}\}$, and $\Lambda = \{(x, gB) \in T^*\mathcal{B} \mid x \in \mathfrak{n}\}$. Then one can give Λ the structure of a scheme as in [K].

Given (Z, Y, k) an admissible triple, let $\Lambda_Y = \Lambda \cap (\mathfrak{n} \times Y)$, $l \geq k$. Then $p_l : Y \to Y_l$ gives rise to the correspondence

$$T^*Y \xleftarrow{\alpha} Y \times_{Y_l} T^*Y_l \xrightarrow{\beta} T^*Y_l$$

and one may check

Lemma 5.1. *β defines an isomorphism between $\alpha^{-1}(\Lambda_Y \cap (\mathfrak{n} \times \mathcal{B}_w))$ and Λ^l_w, for any $\mathcal{B}_w \subseteq Z$.*

Similarly, if $(s, q) \in T \times \mathbf{C}^*$, then one may define its fixpoints on $Y_l, Y, \dots$ and if $n \in \mathfrak{n}$, $s.n = qn$, then we can define $K_*(\mathcal{B}^s_n)$ and an action of $\widehat{H}$ on it, even though the variety $\mathcal{B}^s_n$ need not be rational [KL2].

In the sequel to this paper, we will study these "standard" modules and their irreducible quotients, in the case when G is affine.

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